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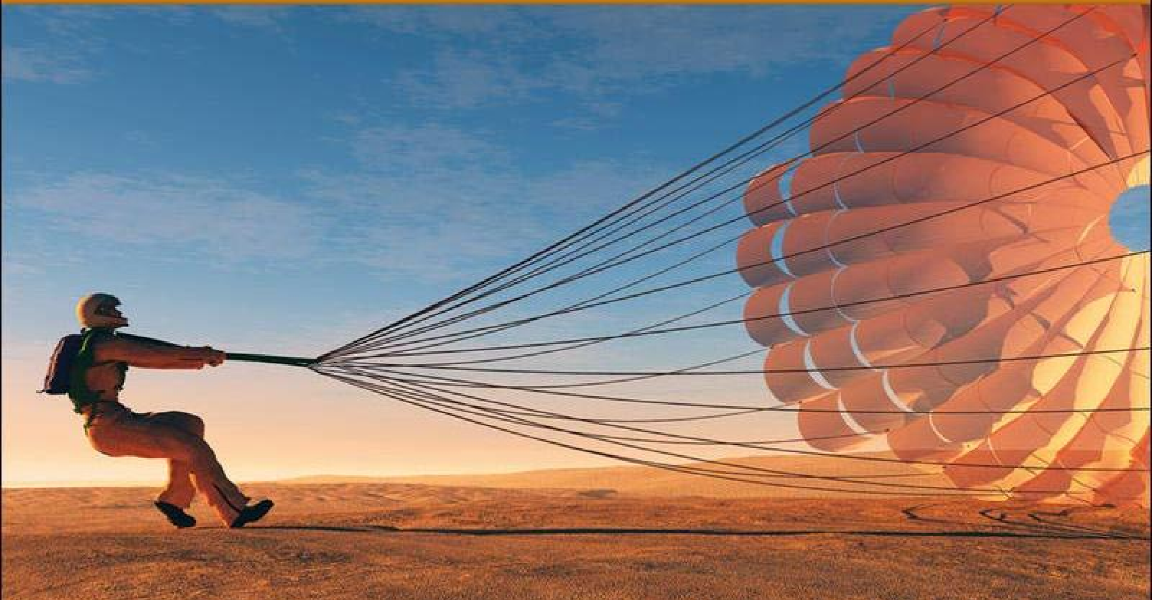
NUMERICAL METHODS FOR ENGINEERS



McGraw Hill

Steven C. Chapra | Raymond P. Canale

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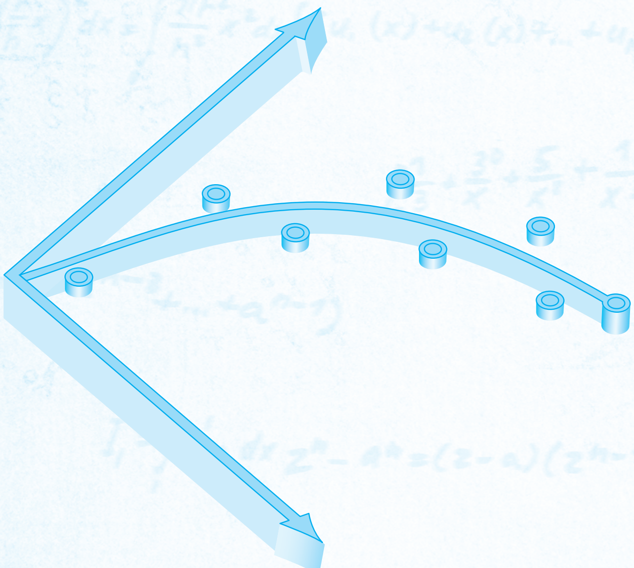

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To

Margaret and Gabriel Chapra

Helen and Chester Canale

ABOUT THE AUTHORS

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Dr. Chapra received engineering degrees from Manhattan College and the University of Michigan. Before joining the faculty at Tufts, he worked for the Environmental Protection Agency and the National Oceanic and Atmospheric Administration, and taught at Texas A&M University, the University of Colorado, and Imperial College London. His general research interests focus on surface water-quality modeling and advanced computer applications in environmental engineering.

He is a Fellow of the ASCE, and has received a number of awards for his scholarly contributions, including the Rudolph Hering Medal (ASCE), and the Meriam-Wiley Distinguished Author Award (American Society for Engineering Education). He has also been recognized as the outstanding teacher among the engineering faculties at Texas A&M University, the University of Colorado, and Tufts University.

Raymond P. Canale is an emeritus professor at the University of Michigan. During his over 20-year career at the university, he taught numerous courses in the areas of computers, numerical methods, and environmental engineering. He also directed extensive research programs in the area of mathematical and computer modeling of aquatic ecosystems. He has authored or coauthored several books and has published over 100 scientific papers and reports. He has also designed and developed personal computer software to facilitate engineering education and the solution of engineering problems. He has been given the Meriam-Wiley Distinguished Author Award by the American Society for Engineering Education for his books and software and several awards for his technical publications.

Professor Canale is now devoting his energies to applied problems, on which he works with engineering firms and industry and governmental agencies as a consultant and expert witness.

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PREFACE

It has been over thirty years since we published the first edition of this book. Over that period, our original contention that numerical methods and computers would figure more prominently in the engineering curriculum—particularly in the early parts—has been dramatically borne out. Many universities now offer freshman, sophomore, and junior courses in both introductory computing and numerical methods. In addition, many of our colleagues are integrating computer-oriented problems into other courses at all levels of the curriculum. Thus, this new edition is still founded on the basic premise that student engineers should be provided with a strong and early introduction to numerical methods. Consequently, we have endeavored to maintain many of the features that made previous editions accessible to both lower- and upper-level undergraduates. These include:

- **Problem Orientation.** Engineering students learn best when they are motivated by problems. This is particularly true for mathematics and computing. Consequently, we have approached numerical methods from a problem-solving perspective.
- **Student-Oriented Pedagogy.** We have developed several features to make this book as student-friendly as possible. These include the overall organization, the use of introductions and epilogues to consolidate major topics and the extensive use of worked examples and case studies from all areas of engineering. We have also endeavored to keep our explanations straightforward and oriented practically.
- **Computational Tools.** We empower our students by helping them utilize the standard "point-and-shoot" numerical problem-solving capabilities of packages like Excel, MATLAB, and Mathcad software. However, students are also shown how to develop simple, well-structured programs to extend the base capabilities of those environments. This knowledge carries over to standard programming languages such as Visual Basic, C/C++, Python, and modern versions of Fortran. We believe that the deemphasis of computer programming represents a "dumbing down" of the engineering curriculum. The bottom line is that if engineers are not content to be tool limited, they will have to write code. Only now they may be called "macros" or "scripts." This book is designed to empower them to do that.

Beyond these original principles, the eighth edition includes new material on cubic splines, Monte Carlo integration, and supplementary material on hyperbolic partial differential equations. It also has new and expanded problem sets. Many of the problems have been modified so that they yield different numerical solutions from previous editions. In addition, a variety of new problems have been included.

As always, our primary intent in writing this book is to provide students with a sound introduction to numerical methods. We believe that motivated students who enjoy numerical methods, computers, and mathematics will, in the end, make better engineers. If our book fosters an enthusiasm for these subjects, we will consider our efforts a success.

Acknowledgments. We would like to express our gratitude to our friends at McGraw-Hill. Special thanks to Heather Ervolino, who provided the positive and supportive atmosphere we needed to create this edition. Louis Poncz and Jane Hoover did a masterful job of copyediting and proofreading the manuscript as did Jeni McAtee as the book's production manager. Last but not least, our marketing manager, Shannon O'Donnell, has ensured that this new edition reaches the widest possible audience of educators and students. As in past editions, David Clough (University of Colorado), Mike Gustafson (Duke), and Jerry Stedinger (Cornell University) generously shared their insights. Useful suggestions were also made by Bill Philpot (Cornell University), Jim Guilkey (University of Utah), Dong-Il Seo (Chungnam National University, Korea), Niall Broekhuizen (NIWA, New Zealand), Marco Pilotti (University of Brescia, Italy), and Raymundo Cordero and Karim Muci (ITESM, Mexico). The text has also benefited from the reviews and suggestions of the following colleagues:

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Scott A. Socolofsky, Texas A&M University

It should be stressed that although we received useful advice from the aforementioned individuals, we are responsible for any inaccuracies or mistakes you may detect in this edition. Please contact Steve Chapra via e-mail if you should detect any errors in this edition.

Finally, we would like to thank our family, friends, and students for their enduring patience and support. In particular, Cynthia Chapra, Danielle Husley, and Claire Canale are always there providing understanding, perspective, and love.

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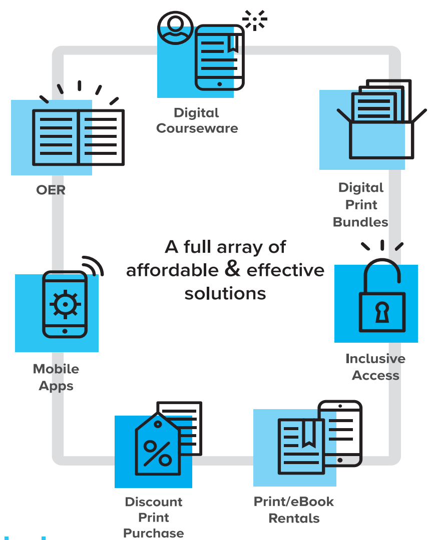

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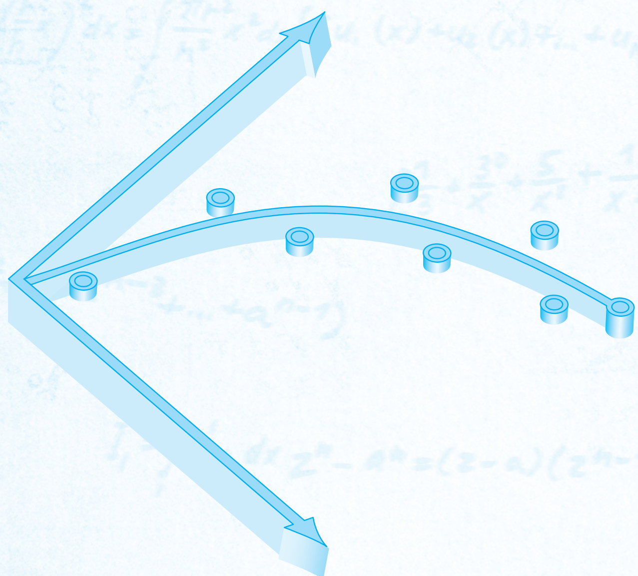
Numerical Methods for Engineers



PART ONE

MODELING, COMPUTERS, AND ERROR ANALYSIS

PT1.1 MOTIVATION

Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations. Although there are many kinds of numerical methods, they have one common characteristic: they invariably involve large numbers of tedious arithmetic calculations. It is little wonder that with the development of fast, efficient digital computers, the role of numerical methods in engineering problem solving has increased dramatically in recent years.

PT1.1.1 Noncomputer Methods

Beyond providing increased computational firepower, the widespread availability of computers (especially personal computers) and their partnership with numerical methods has had a significant influence on the actual engineering problem-solving process. In the precomputer era there were generally three different ways in which engineers approached problem solving:

1. Solutions were derived for some problems using analytical, or exact, methods. These solutions were often useful and provided excellent insight into the behavior of some systems. However, analytical solutions can be derived for only a limited class of problems. These include those that can be approximated with linear models and those that have simple geometry and low dimensionality. Consequently, analytical solutions are of limited practical value because most real problems are nonlinear and involve complex shapes and processes.
2. Graphical solutions were used to characterize the behavior of systems. These graphical solutions usually took the form of plots or nomographs. Although graphical techniques can often be used to solve complex problems, the results are not very precise. Furthermore, graphical solutions (without the aid of computers) are extremely tedious and awkward to implement. Finally, graphical techniques are often limited to problems that can be described using three or fewer dimensions.
3. Calculators and slide rules were used to implement numerical methods manually. Although in theory such approaches should be perfectly adequate for solving complex

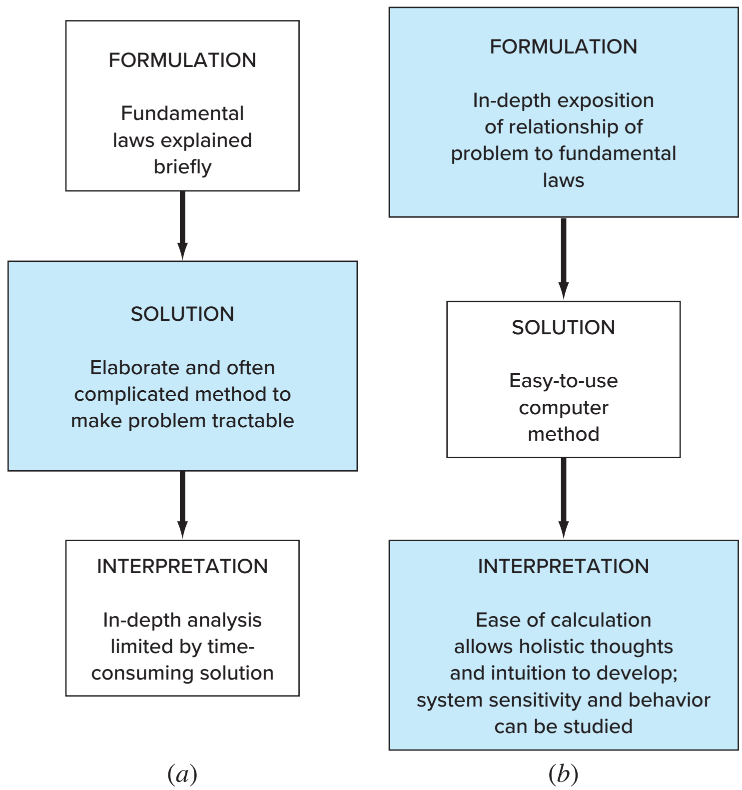


FIGURE PT1.1
The three phases of engineering problem solving in (*a*) the precomputer and (*b*) the computer era. The sizes of the boxes indicate the level of emphasis directed toward each phase. Computers facilitate the implementation of solution techniques and thus allow more emphasis to be placed on the creative aspects of problem formulation and interpretation of results.

problems, in actuality several difficulties are encountered. Manual calculations are slow and tedious. Furthermore, consistent results are elusive because of simple blunders that arise when numerous manual tasks are performed.

During the precomputer era, significant amounts of energy were expended on the solution technique itself, rather than on problem definition and interpretation (Fig. PT1.1*a*). This unfortunate situation existed because so much time and drudgery were required to obtain numerical answers using precomputer techniques.

Today, computers and numerical methods provide an alternative for such complicated calculations. Using computer power to obtain solutions directly, you can approach these calculations without recourse to simplifying assumptions or time-intensive techniques. Although analytical solutions are still extremely valuable both for problem solving and for providing insight, numerical methods represent alternatives that greatly enlarge your capabilities to confront and solve problems. As a result, more time is available for the use of your creative skills. Thus, more emphasis can be placed on problem formulation and solution interpretation and the incorporation of total system, or "holistic," awareness (Fig. PT1.1*b*).

PT1.1.2 Numerical Methods and Engineering Practice

Since the late 1940s the widespread availability of digital computers has led to a veritable explosion in the use and development of numerical methods. At first, this growth was somewhat limited by the cost of access to large mainframe computers, and, consequently, many engineers continued to use simple analytical approaches in a significant portion of their work. Needless to say, the recent evolution of inexpensive personal computers has given us ready access to powerful computational capabilities. There are several additional reasons why you should study numerical methods:

1. Numerical methods are extremely powerful problem-solving tools. They are capable of handling large systems of equations, nonlinearities, and complicated geometries that are not uncommon in engineering practice and that are often impossible to solve analytically. As such, they greatly enhance your problem-solving skills.

2. During your careers, you may often have occasion to use commercially available prepackaged, or "canned," computer programs that involve numerical methods. The intelligent use of these programs is often predicated on knowledge of the basic theory underlying the methods.

3. Many problems cannot be approached using canned programs. If you are conversant with numerical methods and are adept at computer programming, you can design your own programs to solve problems without having to buy or commission expensive software.

4. Numerical methods are an efficient vehicle for learning to use computers. It is well known that an effective way to learn programming is to actually write computer programs. Because numerical methods are for the most part designed for implementation on computers, they are ideal for this purpose. Further, they are especially well-suited to illustrate the power and the limitations of computers. When you successfully implement numerical methods on a computer and then apply them to solve otherwise intractable problems, you will be provided with a dramatic demonstration of how computers can serve your professional development. At the same time, you will also learn to acknowledge and control the errors of approximation that are part and parcel of large-scale numerical calculations.

5. Numerical methods provide a vehicle for you to reinforce your understanding of mathematics. Because one function of numerical methods is to reduce higher mathematics to basic arithmetic operations, they get at the "nuts and bolts" of some otherwise obscure topics. Enhanced understanding and insight can result from this alternative perspective.

PT1.2 MATHEMATICAL BACKGROUND

Every part in this book requires some mathematical background. Consequently, the introductory material for each part includes a section, such as the one you are reading, on mathematical background. Because Part One itself is devoted to background material on mathematics and computers, this section does not involve a review of a specific mathematical topic. Rather, we take this opportunity to introduce you to the types of mathematical subject areas covered in this book. As summarized in Fig. PT1.2, these are

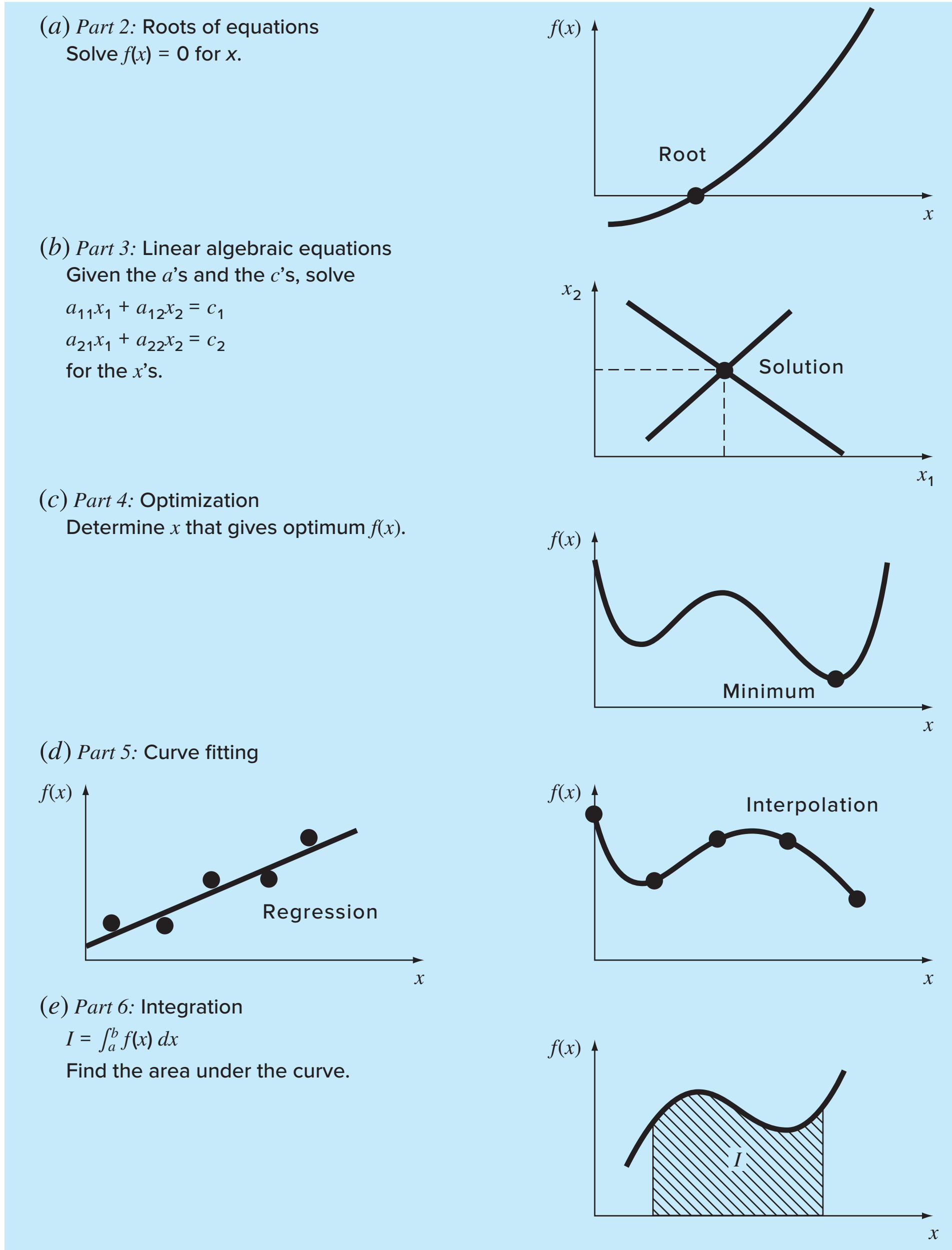


FIGURE PT1.2
Summary of the numerical methods covered in this book.

FIGURE PT1.2 (concluded)

(*f*) *Part 7:* Ordinary differential equations
Given

$$\frac{dy}{dt} \cong \frac{\Delta y}{\Delta t} = f(t, y)$$

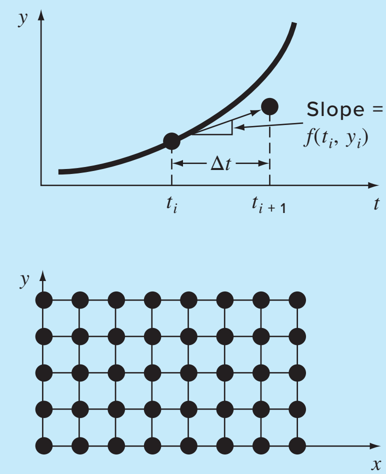
solve for y as a function of t.

$$y_{i+1} = y_i + f(t_i, y_i)\,\Delta t$$

(*g*) *Part 8:* Partial differential equations
Given

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

solve for u as a function of x and y.



1. *Roots of Equations* (Fig. PT1.2*a*). These problems are concerned with the value of a variable or a parameter that satisfies a single nonlinear equation. These problems are especially valuable in engineering design contexts where it is often impossible to explicitly solve design equations for parameters.
2. *Systems of Linear Algebraic Equations* (Fig. PT1.2*b*). These problems are similar in spirit to roots of equations in the sense that they are concerned with values that satisfy equations. However, in contrast to satisfying a single equation, a set of values is sought that simultaneously satisfies a set of linear algebraic equations. Such equations arise in a variety of problem contexts and in all disciplines of engineering. In particular, they originate in the mathematical modeling of large systems of interconnected elements such as structures, electric circuits, and fluid networks. However, they are also encountered in other areas of numerical methods such as curve fitting and differential equations.
3. *Optimization* (Fig. PT1.2*c*). These problems involve determining a value or values of an independent variable that correspond to a "best," or optimal, value of a function. Thus, as in Fig. PT1.2*c*, optimization involves identifying maxima and minima. Such problems occur routinely in engineering design contexts. They also arise in a number of other numerical methods. We address both single- and multivariable unconstrained optimization. We also describe constrained optimization with particular emphasis on linear programming.
4. *Curve Fitting* (Fig. PT1.2*d*). You will often have occasion to fit curves to data points. The techniques developed for this purpose can be divided into two general categories: regression and interpolation. Regression is employed where there is a significant degree of error associated with the data. Experimental results are often of this kind. For these situations, the strategy is to derive a single curve that represents the general trend of the data without necessarily matching any individual points. In contrast,

interpolation is used where the objective is to determine intermediate values between relatively error-free data points. Such is usually the case for tabulated information. For these situations, the strategy is to fit a curve directly through the data points and use the curve to predict the intermediate values.

5. *Integration* (Fig. PT1.2*e*). As depicted, a physical interpretation of numerical integration is the determination of the area under a curve. Integration has many applications in engineering practice, ranging from the determination of the centroids of oddly shaped objects to the calculation of total quantities based on sets of discrete measurements. In addition, numerical integration formulas play an important role in the solution of differential equations.
6. *Ordinary Differential Equations* (Fig. PT1.2*f*). Ordinary differential equations are of great significance in engineering practice. This is because many physical laws are couched in terms of the rate of change of a quantity rather than the magnitude of the quantity itself. Examples range from population-forecasting models (rate of change of population) to the acceleration of a falling body (rate of change of velocity). Two types of problems are addressed: initial-value and boundary-value problems. In addition, the computation of eigenvalues is covered.
7. *Partial Differential Equations* (Fig. PT1.2*g*). Partial differential equations are used to characterize engineering systems where the behavior of a physical quantity is couched in terms of its rate of change with respect to two or more independent variables. Examples include the steady-state distribution of temperature on a heated plate (two spatial dimensions) or the time-variable temperature of a heated rod (time and one spatial dimension). Two fundamentally different approaches are employed to solve partial differential equations numerically. In the present text, we will emphasize finite-difference methods that approximate the solution in a pointwise fashion (Fig. PT1.2*g*). However, we will also present an introduction to finite-element methods, which use a piecewise approach.

PT1.3 ORIENTATION

Some orientation might be helpful before proceeding with our introduction to numerical methods. The following is intended as an overview of the material in Part One. In addition, some objectives have been included to focus your efforts when studying the material.

PT1.3.1 Scope and Preview

Figure PT1.3 is a schematic representation of the material in Part One. We have designed this diagram to provide you with a global overview of this part of the book. We believe that a sense of the "big picture" is critical to developing insight into numerical methods. When reading a text, it is often possible to become lost in technical details. Whenever you feel that you are losing the big picture, refer back to Fig. PT1.3 to reorient yourself. Every part of this book includes a similar figure.

Figure PT1.3 also serves as a brief preview of the material covered in Part One. *Chapter 1* is designed to orient you to numerical methods and to provide motivation by demonstrating how these techniques can be used in the engineering modeling process.

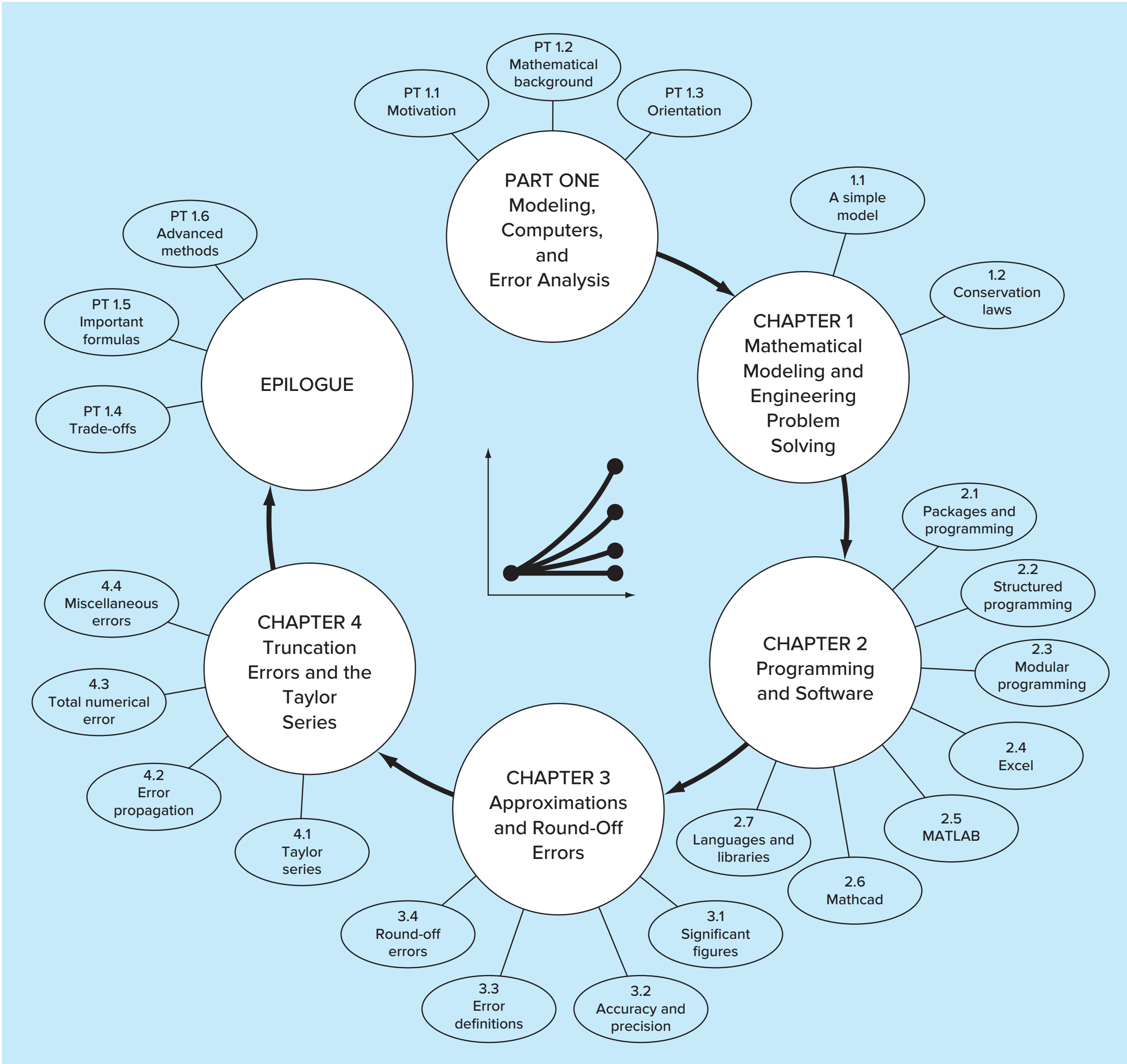


FIGURE PT1.3
Schematic of the organization of the material in Part One: Modeling, Computers, and Error Analysis.

TABLE PT1.1 Specific study objectives for Part One.

1. Recognize the difference between analytical and numerical solutions.
2. Understand how conservation laws are employed to develop mathematical models of physical systems.
3. Define top-down and modular design.
4. Delineate the rules that underlie structured programming.
5. Be capable of composing structured and modular programs in a high-level computer language.
6. Know how to translate structured flowcharts and pseudocode into code in a high-level language.
7. Start to familiarize yourself with any software packages that you will be using in conjunction with this text.
8. Recognize the distinction between truncation and round-off errors.
9. Understand the concepts of significant figures, accuracy, and precision.
10. Recognize the difference between true relative error ε_t, approximate relative error ε_a, and acceptable error ε_s, and understand how ε_a and ε_s are used to terminate an iterative computation.
11. Understand how numbers are represented in digital computers and how this representation induces round-off error. In particular, know the difference between single and extended precision.
12. Recognize how computer arithmetic can introduce and amplify round-off errors in calculations. In particular, appreciate the problem of subtractive cancellation.
13. Understand how the Taylor series and its remainder are employed to represent continuous functions.
14. Know the relationship between finite divided differences and derivatives.
15. Be able to analyze how errors are propagated through functional relationships.
16. Be familiar with the concepts of stability and condition.
17. Familiarize yourself with the trade-offs outlined in the Epilogue of Part One.

Chapter 2 is an introduction and review of computer-related aspects of numerical methods and suggests the level of computer skills you should acquire to efficiently apply succeeding information. *Chapters 3* and *4* deal with the important topic of error analysis, which must be understood for the effective use of numerical methods. In addition, an *epilogue* is included that introduces the trade-offs that have such great significance for the effective implementation of numerical methods.

PT1.3.2 Goals and Objectives

Study Objectives. Upon completing Part One, you should be adequately prepared to embark on your studies of numerical methods. In general, you should have gained a fundamental understanding of the importance of computers and the role of approximations and errors in the implementation and development of numerical methods. In addition to these general goals, you should have mastered each of the specific study objectives listed in Table PT1.1.

Computer Objectives. Upon completing Part One, you should have mastered sufficient computer skills to develop your own software for the numerical methods in this text. You should be able to develop well-structured and reliable computer programs on the basis of pseudocode, flowcharts, or other forms of algorithms. You should have developed the capability to document your programs so that they may be effectively employed by users. Finally, in addition to your own programs, you may be using software packages along with this book. Packages like Excel, Mathcad, or The MathWorks, Inc. MATLAB® program are examples of such software. You should become familiar with these packages, so that you will be comfortable using them to solve numerical problems later in the text.

CHAPTER

1

Mathematical Modeling and Engineering Problem Solving

Knowledge and understanding are prerequisites for the effective implementation of any tool. No matter how impressive your tool chest, you will be hard-pressed to repair a car if you do not understand how it works.

This is particularly true when using computers to solve engineering problems. Although they have great potential utility, computers are practically useless without a fundamental understanding of how engineering systems work.

This understanding is initially gained by empirical means—that is, by observation and experiment. However, while such empirically derived information is essential, it is only half the story. Over years and years of observation and experiment, engineers and scientists have noticed that certain aspects of their empirical studies occur repeatedly. Such general behavior can then be expressed as fundamental laws that essentially embody the cumulative wisdom of past experience. Thus, most engineering problem solving employs the two-pronged approach of empiricism and theoretical analysis (Fig. 1.1).

It must be stressed that the two prongs are closely coupled. As new measurements are taken, the generalizations may be modified or new ones developed. Similarly, the generalizations can have a strong influence on the experiments and observations. In particular, generalizations can serve as organizing principles that can be employed to synthesize observations and experimental results into a coherent and comprehensive framework from which conclusions can be drawn. From an engineering problem-solving perspective, such a framework is most useful when it is expressed in the form of a mathematical model.

The primary objective of this chapter is to introduce you to mathematical modeling and its role in engineering problem solving. We will also illustrate how numerical methods figure in the process.

1.1 A SIMPLE MATHEMATICAL MODEL

A mathematical model can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms. In a very general sense, it can be represented as a functional relationship of the form

$$\begin{matrix}\text{Dependent}\\ \text{variable}\end{matrix} = f\left(\begin{matrix}\text{independent}\\ \text{variables}\end{matrix}, \text{parameters}, \begin{matrix}\text{forcing}\\ \text{functions}\end{matrix}\right) \tag{1.1}$$

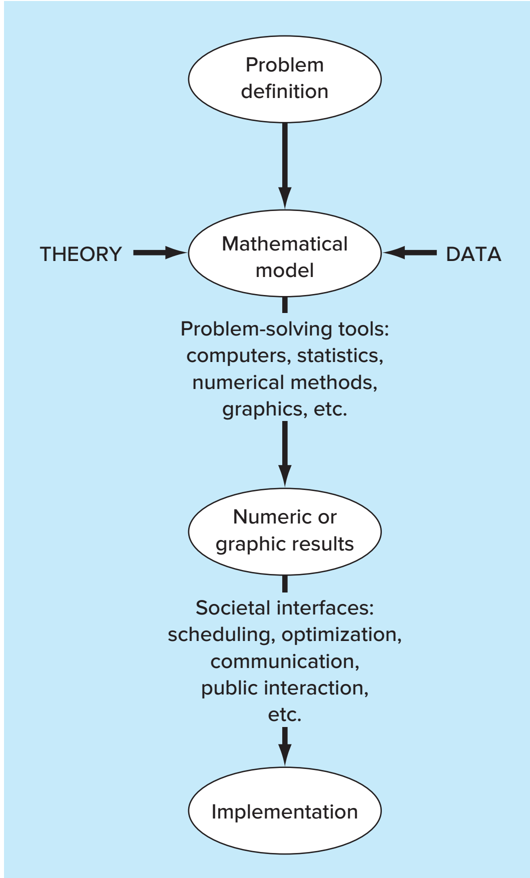


FIGURE 1.1
The engineering problem-solving process.

where the *dependent variable* is a characteristic that usually reflects the behavior or state of the system; the *independent variables* are usually dimensions, such as time and space, along which the system's behavior is being determined; the *parameters* are reflective of the system's properties or composition; and the *forcing functions* are external influences acting upon the system.

The actual mathematical expression of Eq. (1.1) can range from a simple algebraic relationship to large complicated sets of differential equations. For example, on the basis of his observations, Newton formulated his second law of motion, which states that the time rate of change of momentum of a body is equal to the resultant force acting on it. The mathematical expression, or model, of the second law is the well-known equation

$$F = ma \tag{1.2}$$

where F = net force acting on the body (N, or kg m/s^2), m = mass of the object (kg), and a = its acceleration (m/s^2).